Using this formula, the average gain for any valid antenna pattern with elevation gain that is maximum at $\varepsilon = 0$, is symmetric about $\varepsilon = 0$, and decreases monotonically for $|\varepsilon| > 0$, will always be less than 0 dB (which is the result for an isotropic antenna). This can be seen by observing that the total power radiated through the portion of the upper hemisphere for which $\varepsilon \geq 0$ is:

$$P(\varepsilon) = \frac{P}{4\pi} \int_{-\pi}^{\pi} \int_{\varepsilon_0}^{\pi/2} G(\alpha, \varepsilon) \cos\varepsilon \, d\varepsilon \, d\alpha \ . \tag{8}$$

With the gain antenna, more power is directed at the elevation range excluded by the integral ($\varepsilon < \varepsilon_0$) than with the isotropic antenna, so there is less total power directed toward the upper portion of the hemisphere ($\varepsilon \geq \varepsilon_0$). The integral in (7) is therefore less than in the isotropic case for antennas with horizonal gain.

Evaluation of (7) using the AirTouch antenna pattern of (5), corrected to satisfy (2), bears this out, giving the results shown in Table 1.

Table 1: Corrected AirTouch Results

B_ε (degrees)	G_{av} (dBi)
10	-3.39
20	-1.55
30	-0.85
40	-0.53
50	-0.35
60	-0.24
70	-0.18

THE PDF OF THE ELEVATION ANGLE

The portion of the Earth's surface seen by the satellite subtends a polar angle β_{max} (see Fig. 1), where the polar axis extends from the satellite to the center of the Earth. An area element can be expressed as $dA = R\,d\beta \cdot R\sin\beta\, d\alpha$. Assuming that terrestrial transmitters are uniformly-distributed over area, the probability that a given transmitter is located within an angular increment $d\beta$ at an angle β from the axis is proportional to $\sin\beta$. Applying the requirement that the pdf must integrate to 1 gives:

$$f_\beta(\beta) = \frac{\sin\beta}{1 - \cos\beta_{max}}, \quad 0 \leq \beta \leq \beta_{max} \ . \tag{9}$$

The elevation angle ε can be related to β using Figure 3. The law of cosines gives:

$$d = \sqrt{R^2 \sin^2 \varepsilon + 2Rh + h^2} - R \sin \varepsilon. \tag{10}$$

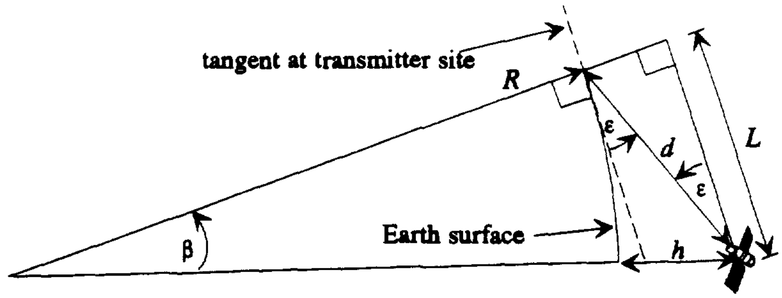

Figure 3: *Geometry for relating θ to ε*

From Fig. 3, $L = (R+h)\sin\beta = d\cos\varepsilon$. Hence, $\sin\beta = d\cos\varepsilon/(R+h)$, which with (10) gives β as a function of ε. Defining $u \equiv \sin\beta$, transforming (9) gives the pdf of u as:

$$f_u(u) = \frac{u}{(1-\cos\beta_{\max})\sqrt{1-u^2}}, \quad 0 \le u \le \sin\beta_{\max}. \tag{11}$$

The pdf of ε is then $f_\varepsilon(\varepsilon) = -f_u\left(\frac{d\cos\varepsilon}{R+h}\right)\frac{du}{d\varepsilon}$, with

$$\frac{du}{d\varepsilon} = -\frac{d}{R+h}\left[\frac{R\cos^2\varepsilon}{d+R\sin\varepsilon} + \sin\varepsilon\right]. \tag{12}$$

Figure 4 shows $f_\varepsilon(\varepsilon)$ and $\cos\varepsilon/(1-\sin\varepsilon_0)$ for $\varepsilon_0 = 10°$. Notice that $f_\varepsilon(\varepsilon)$ weights low elevation angles much more heavily than does $\cos\varepsilon/(1-\sin\varepsilon_0)$, so it can be expected to yield higher average gain for antennas with elevation gain.

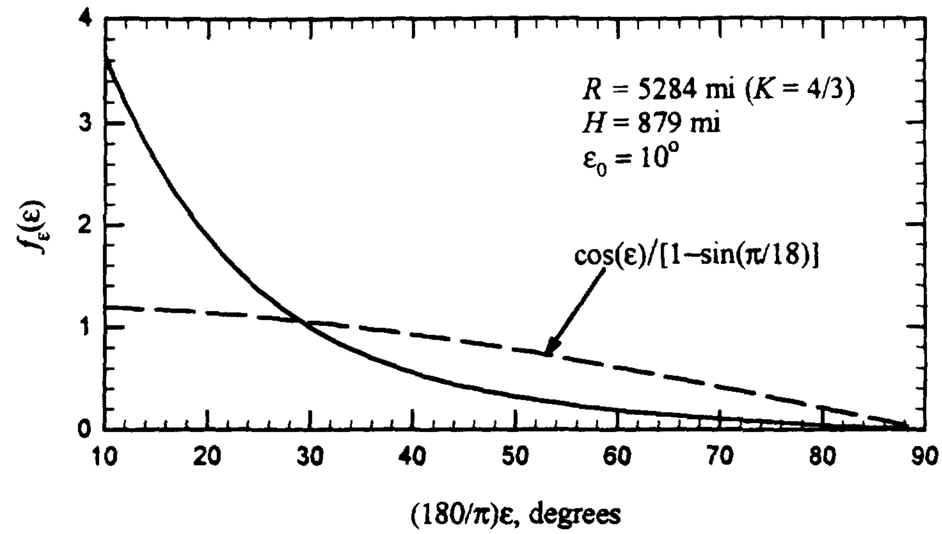

Figure 4: *Probability density function of the elevation angle, assuming terrestrial transmitters are uniformly-distributed over area.*

AVERAGE GAIN RESULTS FOR SEVERAL ANTENNAS

In this section, the average gain will be computed, using the "uniform density" pdf for the elevation angle developed above, for three different antenna types: (1) the adjusted "AirTouch" antenna discussed above; (2) parabolic dish antennas of various diameters; and (3) a half-wave dipole.

The AirTouch Antenna

Table 2 shows the average gain for the AirTouch antenna, computed using $f_\varepsilon(\varepsilon)$ instead of $\cos\varepsilon/(1-\sin\varepsilon_0)$. Note that in this case, there is an increase in average gain (albeit very small) over the isotropic case, due to the heavier weighting of lower elevation angles. The maximum is about 0.6 dB, occurring at elevation beamwidths of 30 to 40 degrees.

Table 2: *Average gain for the AirTouch antenna pattern using the "uniform density" probability density function for the elevation angle.*

B_ε (degrees)	G_{av} (dBi)
5.000	-4.381
10.000	-2.115
15.000	-0.433
20.000	0.253
25.000	0.519
30.000	0.604
35.000	0.606
40.000	0.571
45.000	0.521
50.000	0.467
55.000	0.415
60.000	0.368
65.000	0.325
70.000	0.288
75.000	0.255

Parabolic Dish Antennas

The circular parabolic reflector is an important antenna to consider, because near 5 GHz, considerable gain can be achieved with a small antenna. It is reasonable to expect that for NII/SUPERNet applications requiring antenna gain, the parabolic dish will often be the antenna of choice.

If ϕ is the polar angle between the field point and the antenna boresight (see Fig. 2), the normalized field pattern is:[1]

$$E(\phi) = \frac{2}{\pi D_\lambda} \frac{J_1(\pi D_\lambda \sin\phi)}{\sin\phi}, \tag{13}$$

where D_λ is the diameter of the aperture in wavelengths and $J_1(\cdot)$ is the first-order Bessel function of the first kind. Note that $E(\phi)_{max} = E(0) = 1$,[2] so the power gain pattern can be written as:

[1] John D. Krauss, *Antennas*, second edition, McGraw-Hill, 1988, section 12.7.

[2] $\lim_{x \to 0} J_1(x) = x/2$

$$G(\phi) = G_0 E^2(\phi) = G_0 \left[\frac{2}{\pi D_\lambda} \frac{J_1(\pi D_\lambda \sin\phi)}{\sin\phi} \right]^2 \tag{14}$$

Figure 5 shows power gain patterns for several values of $D_\lambda = 2$, 5, and 10.

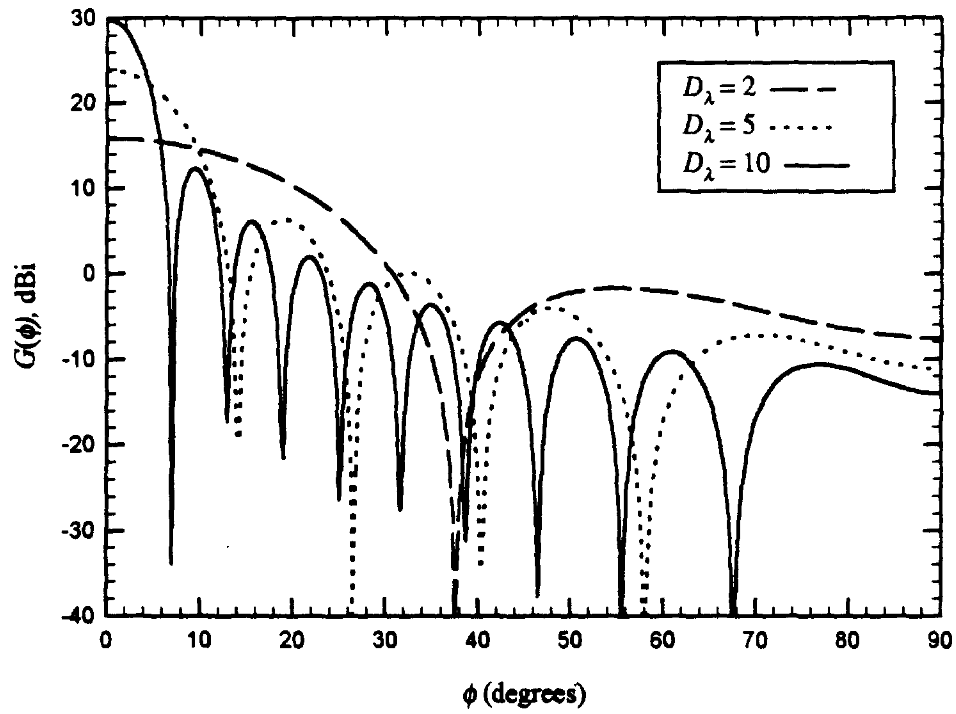

Figure 5: *Gain pattern for parabolic dish antenna; D_λ is the diameter in wavelengths.*

The average gain can be computed using (4) with (see Fig. 2) $\phi = \cos^{-1}(\cos\varepsilon \cdot \cos\alpha)$. Assuming that all of the power is directed in the hemisphere $-\pi/2 \le \alpha \le \pi/2$, the average gain can be calculated as:

$$G_{av} = \frac{1}{\pi} \int_0^{\pi/2} \int_{\varepsilon_0}^{\pi/2} G\left[\cos^{-1}(\cos\alpha\cos\varepsilon)\right] f_\varepsilon(\varepsilon) d\varepsilon d\alpha \; . \tag{15}$$

Table 3 shows the average gain and other parameters of interest for parabolic dish antennas with D_λ ranging from 1 to 10. The beamwidths are in degrees. The maximum gain was computed using:

$$G_0 = \frac{\pi}{\int_0^{\pi/2} \int_{\varepsilon_0}^{\pi/2} E^2\left[\cos^{-1}(\cos\alpha\cos\varepsilon)\right] \cos\varepsilon \, d\varepsilon d\alpha}, \tag{16}$$

and agrees well with the theoretical directive gain for a large, uniformly-illuminated aperture:[3] $G = 9.87D_\lambda^2$; note that agreement is closer as the aperture size increases. As with the AirTouch antenna formula in (5), the average gain decreases as the maximum gain increases, since more power is directed below the minimum elevation angle seen by the satellite.

Table 3: *Beamwidths, maximum gain, and average gain for parabolic dish antennas using the "uniform density" pdf for the elevation angle.*

D_λ	half-power beamwidth	first null beamwidth	maximum gain (dBi)	average gain (dBi)
1.000	58.000	140.000	9.659	.747
2.000	29.000	70.000	15.858	.097
3.000	19.333	46.667	19.427	-1.836
4.000	14.500	35.000	21.946	-4.123
5.000	11.600	28.000	23.895	-5.806
6.000	9.667	23.333	25.485	-6.397
7.000	8.286	20.000	26.828	-6.542
8.000	7.250	17.500	27.991	-6.898
9.000	6.444	15.556	29.016	-7.577
10.000	5.800	14.000	29.933	-8.343

The Half-Wave Dipole

For the coordinate system used here, the pattern of a vertical (normal to the Earth's surface) half-wave dipole can be written as:

$$G_{\lambda/2}(\varepsilon) = G_0 \left[\frac{\cos\left(\frac{\pi}{2}\sin\varepsilon\right)}{\cos\varepsilon} \right]^2 \tag{17}$$

The pattern is omnidirectional in azimuth; hence:

$$\int_0^{\pi/2} G_{\lambda/2}(\varepsilon)\cos\varepsilon\, d\varepsilon = 1, \tag{18}$$

[3] Krauss, *op. cit.*

which allows G_0 to be computed numerically and compared to the well-known value of 1.64 (2.15 dBi) as a sanity check on the numerical computations. Table 4 shows the average gain computed using (4), as well as using the AirTouch formula (7).

Table 4: *Average gain for a half-wave dipole, using the "uniform density" pdf for the elevation angle, and also using the AirTouch formula*

minimum elevation angle (deg.)	average gain (uniforum density), dBi	average gain (AirTouch formula), dBi
0	1.448	0.000
2	1.345	-0.102
4	1.226	-0.213
6	1.091	-0.333
8	0.941	-0.463
10	**0.776**	**-0.603**
12	0.596	-0.753
14	0.401	-0.914
16	0.191	-1.085
18	-0.034	-1.267
20	-0.272	-1.461

As can be seen, for a minimum elevation angle greater than zero, the AirTouch average gain formula yields a result less than 0 dBi, because the dipole has gain in the elevation dimension. The computing the average gain using the "uniform density" pdf for ε gives a larger average gain (as would be expected), but with a 10° minimum elevation angle, the average is less than 0.8 dBi. However, this is larger than the average gain for either the parabolic dish or the AirTouch antenna pattern under any of the conditions investigated here.

CONCLUSIONS

It is clear from the derivations and examples given here that, consistent with intuition, the potential for interference from unlicensed devices to the MSS feeder uplink at 5150-5250 MHz will be determined not by the average EIRP of the unlicensed devices, but rather by the average transmit power. The maximum average gain observed in the calculations given here was for the half-wave dipole (less than 0.8 dBi), which has a maximum gain of about 2.15 dBi. The average gain of "high-gain" antennas (such as a parabolic dish) is less than 0 dBi, because much of the power is directed below the satellite's field of vision. For the average gain computations discussed here, all antennas were assumed to be of the same type. In reality, there will be a mix of antenna types. If these include high-gain antennas, then the results given here suggest that the average gain over all antennas will be less than 0 dBi.

It is recommended that the power output limit for unlicensed devices operating in the 5150-5250 MHz band be stated as a limit on power into the antenna terminals, rather than as an EIRP limit. This approach would obviously benefit the unlicensed devices, allowing design flexibility and simplifying compliance measurements. However, it would benefit the MSS feeder uplinks as well. A transmit power limit would encourage the designers of the unlicensed devices to use directional antennas when it would benefit the applications. As shown here, the average gain (seen by the satellite) for directional antennas is actually less than that for antennas that are more nearly isotropic. Thus, a transmitter output power limit, rather than an EIRP limit, may actually *reduce* interference to MSS feeder uplinks.

APPENDIX

EXCESS GAIN OF THE AIRTOUCH ANTENNA PATTERN

The AirTouch antenna pattern is:

$$G_{AT}(\alpha,\varepsilon)=\frac{27000}{B_\alpha B_\varepsilon}10^{-\frac{1}{2}\left[\left(\frac{\varepsilon}{B_\varepsilon}\right)^2+\left(\frac{\alpha}{B_\alpha}\right)^2\right]}+1, \tag{A-1}$$

which can be written as:

$$G_{AT}(\alpha,\varepsilon)=27000\frac{e^{-K_\alpha^2\alpha^2}}{B_\alpha}\cdot\frac{e^{-K_\varepsilon^2\varepsilon^2}}{B_\varepsilon}+1, \tag{A-2}$$

where $K_\alpha=\frac{\sqrt{\ln 10/2}}{B_\alpha}$ and $K_\varepsilon=\frac{\sqrt{\ln 10/2}}{B_\varepsilon}$.

The conservation-of-energy constraint on the gain pattern is:

$$\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}G(\alpha,\varepsilon)\cos\varepsilon\, d\varepsilon d\alpha=4\pi \tag{A-3}$$

Since the AirTouch gain pattern is symmetric about $\alpha=0$ and about $\varepsilon=0$, the excess gain can be expressed as:

$$G_{excess}=\frac{1}{\pi}\int_0^{\pi}\int_0^{\pi/2}G_{AT}(\alpha,\varepsilon)\cos\varepsilon\, d\varepsilon d\alpha=1+\frac{27000}{\pi}\left(\frac{\pi}{180}\right)^2 I_\alpha I_\varepsilon, \tag{A-4}$$

where the factor $(\pi/180)^2$ accounts for the conversion from degrees to radians, and:

$$I_\alpha=\frac{1}{B_\alpha}\int_0^{\pi}e^{-K_\alpha^2\alpha^2}\,d\alpha \tag{A-5a}$$

$$I_\varepsilon=\frac{1}{B_\varepsilon}\int_0^{\pi/2}e^{-K_\varepsilon^2\varepsilon^2}\cos\varepsilon\, d\varepsilon \tag{A-5b}$$

Using the substitution $x = K_\alpha \alpha$, (A-5a) becomes:

$$I_\alpha = \sqrt{\frac{2}{\ln 10}} \int_0^{\pi K_\alpha} e^{-x^2} dx = \sqrt{\frac{\pi}{2\ln 10}} \cdot \mathrm{erf}\left(\frac{\pi}{B_\alpha}\sqrt{\frac{\ln 10}{2}}\right), \tag{A-6}$$

where $\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2} dt$ is the error function. Unless B_α is large (i.e., $\geq \pi/2$), $\mathrm{erf}\left(\pi/B_\alpha \sqrt{\ln 10/2}\right) \approx 1$, and

$$I_\alpha \approx \sqrt{\frac{\pi}{2\ln 10}}. \tag{A-7}$$

Similarly, substituting $x = K_\varepsilon \varepsilon$ in (A-5b) gives:

$$I_\varepsilon = \sqrt{\frac{2}{\ln 10}} \int_0^{\pi K_\varepsilon/2} e^{-x^2} \cos(x/K_\varepsilon) d\varepsilon \tag{A-8}$$

Using $\int_0^\infty e^{-x^2} \cos ax\, dx = \frac{\sqrt{\pi}}{2} e^{-a^2/4}$, and assuming that $B_\varepsilon < \pi/2$, I_ε can be approximated as:

$$I_\varepsilon \approx \sqrt{\frac{\pi}{2\ln 10}} e^{-B_\varepsilon^2/2\ln 10} \tag{A-9}$$

Combining (A-4), (A-7), and (A-9), the excess gain can be approximated as:

$$G_{excess} \approx 1 + \left(\frac{\pi}{180}\right)^2 \frac{27000}{2\ln 10} e^{-B_\varepsilon^2/2\ln 10} = 1 + 1.786 e^{-B_\varepsilon^2/2\ln 10}, \tag{A-10}$$

where B_ε is in radians. The approximation of (A-10) was checked against numerical results, and found to be accurate to within less than 0.1 dB for $B_\varepsilon \leq 75°$ and $B_\alpha \leq 120°$. For $B_\varepsilon \leq 50°$ and $B_\alpha \leq 80°$, it agrees with the numerical results to within 0.01 dB. The largest error observed for any combination of B_ε and B_α was less than 0.4 dB for $0 < B_\alpha \leq 180°$ and $0 < B_\varepsilon \leq 90°$.